MATH 2010B Advanced Calculus I Lecture Notes Week 10

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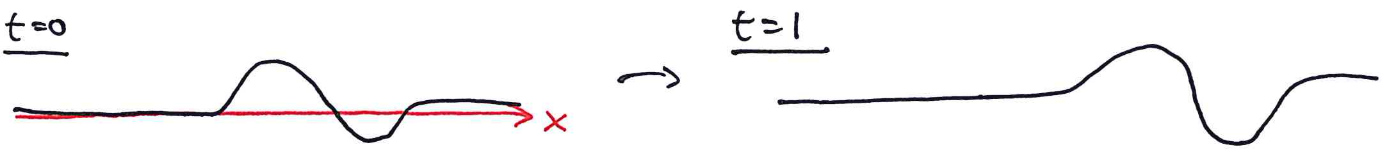
Last time ... Chain Rule, implicit differentiation and change of variable.

Physical Application (1D Wave Equation)

Setup: We have an infinitely long string



Define a function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(t,x) = \text{height of the string at position } x \text{ and time } t.$$

The governing law of equation is

wave equation: $$\boxed{f_{tt} = f_{xx}} \quad \text{——— } (*)$$

Q: What are the possible solutions f for $(*)$?

To solve $(*)$, we introduce new variables:

$$\begin{cases} u = x + t \\ v = x - t \end{cases} \qquad f(x,y) = f(u,v)$$

Q: What equation corresponding to $(*)$ in the new variables?

→ Chain rule helps!

eg:
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\underbrace{\left(\frac{\partial u}{\partial x}\right)}_{=1} + \frac{\partial f}{\partial v}\underbrace{\left(\frac{\partial v}{\partial x}\right)}_{=1}.$$

$$\Rightarrow \quad f_x = f_u + f_v$$

differentiatiate w.r.t x again,

$$f_{xx} = \frac{\partial}{\partial x} f_u + \frac{\partial}{\partial x} f_v$$

$$= \left(\frac{\partial f_u}{\partial u}\underbrace{\frac{\partial u}{\partial x}}_{=1} + \frac{\partial f_u}{\partial v}\underbrace{\frac{\partial v}{\partial x}}_{=1}\right) + \left(\frac{\partial f_v}{\partial u}\underbrace{\frac{\partial u}{\partial x}}_{=1} + \frac{\partial f_v}{\partial v}\underbrace{\frac{\partial v}{\partial x}}_{=1}\right)$$

[Assume $f \in C^2$. so $f_{uv} = f_{vu}$]

$$= (f_{uu} + f_{uv}) + (f_{vu} + f_{vv})$$

$$= f_{uu} + f_{vv} + 2 f_{uv}$$

Similarly,

$$f_t = f_u \underbrace{\frac{\partial u}{\partial t}}_{=1} + f_v \underbrace{\frac{\partial v}{\partial t}}_{=-1} = f_u - f_v$$

$$f_{tt} = \left(f_{uu}\underbrace{\frac{\partial u}{\partial t}}_{=1} + f_{uv}\underbrace{\frac{\partial v}{\partial t}}_{=-1}\right) - \left(f_{vu}\underbrace{\frac{\partial u}{\partial t}}_{=1} + f_{vv}\underbrace{\frac{\partial v}{\partial t}}_{=-1}\right)$$

$$= f_{uu} - f_{uv} - f_{vu} + f_{vv}$$

$$= f_{uu} + f_{vv} - 2 f_{uv}.$$

(*) $\boxed{f_{tt} = f_{xx}}$ $\iff$ $\cancel{f_{uu}} + \cancel{f_{vv}} - 2 f_{uv} = \cancel{f_{uu}} + \cancel{f_{vv}} + 2 f_{uv}$

$\iff$ (**) $\boxed{f_{uv} = 0}$ wave equation in u, v variable

Look at (**), $(f_u)_v = 0$

$\Rightarrow$ $f_u(u,v) = F(u)$ for some function $F(u)$ depending only on u.

$\Rightarrow$ $\dfrac{\partial f}{\partial u} = F(u)$

integrate w.r.t. u $\Rightarrow$ $f = \int F(u)\,du + G(v)$ for some function $G(v)$ depending only on v.

$f(u,v) = F(u) + G(v)$

This is special becuz $f(u,v) = uv \neq F(u) + G(v)$

Back to x, t variable, solutions to equation (*) have the form

$$\boxed{f(t,x) = F(x+t) + G(x-t)} \quad \text{—(\#)}.$$

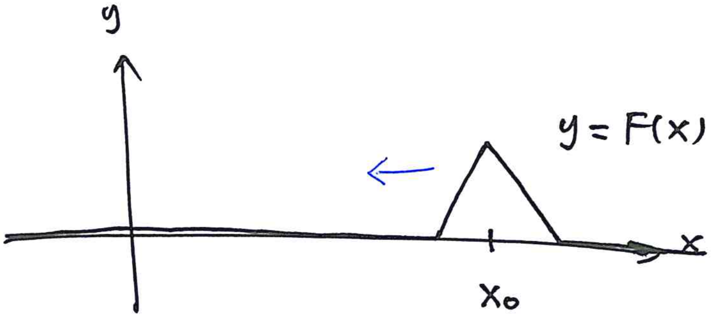
for some 1-variable functions F & G.

$\underline{Q}$: What does (#) tell us physically?

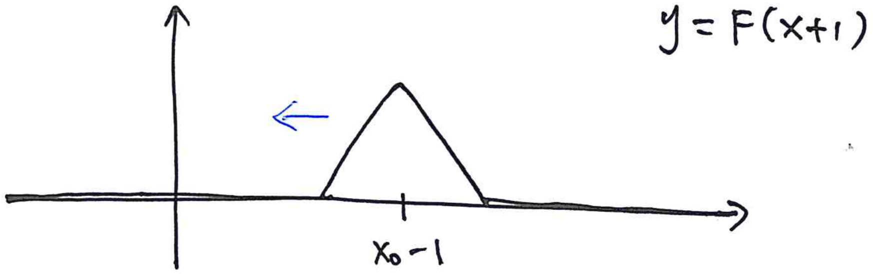
$\underline{1^{st}}$: Let $G \equiv 0$. so $f(t,x) = F(x+t)$.

- This represents a wave travelling to the left at speed $= 1$.

$\underline{t=0}$:



$\underline{t=1}$:

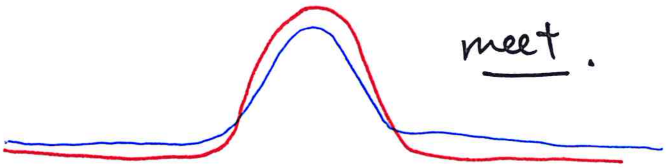


Similarly. $$f(t,x) = \underbrace{F(x+t)}_{\text{a wave travelling to the left at speed} = 1} + \underbrace{G(x-t)}_{\text{a wave travelling to the right at speed} = 1}.$$

"law of superposition".

t=0:

t=1:



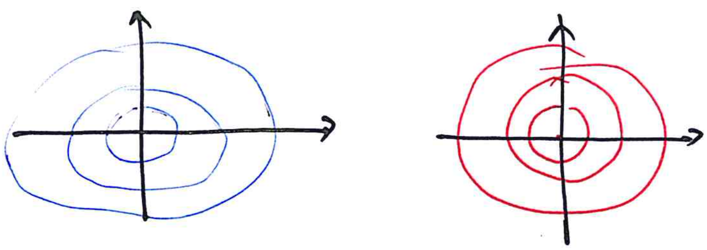
meet.

$=$ add up $\leftrightarrow$ "superposition"

t=2:

Note: There are higher dimensional wave equations.

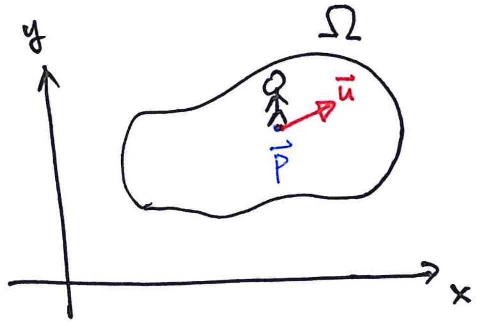
(2D) $$u_{tt} = u_{xx} + u_{yy}.$$



Last time ... chain rule, 1D wave equation ...

Directional Derivative

Physical Problem A:



$f : \Omega \to \mathbb{R}$

$f(x,y)$ = temperature at $(x,y) = \vec{p}$

Q: which direction $\vec{u}$ should 웃 go to get warmer in the fastest way?

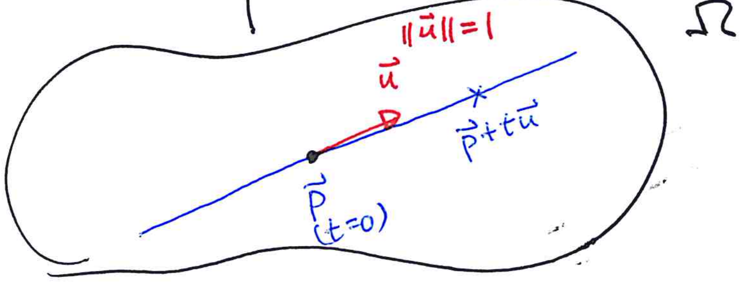
$\Rightarrow$ need the concept of "rate of change of f along some direction"

"Direction": $\vec{u} \in \mathbb{R}^2$, unit vector $\|\vec{u}\| = 1$.

Define:
$$\boxed{D_{\vec{u}} f(\vec{p}) := \lim_{t \to 0} \frac{f(\vec{p} + t\vec{u}) - f(\vec{p})}{t}}$$

Directional Derivative of f at $\vec{p}$ along the direction $\vec{u}$

diff. f restricted on the blue line (at $t=0$)



E.g. 1: Calculate $D_{\vec{u}} f(\vec{p})$ where $f(x,y) = x^2$, $\vec{p} = (1,0)$ and $\underbrace{\vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}_{\|\vec{u}\| = 1}$.

Sol: Parametrize the line : $\vec{p} + t\vec{u} = \left(1+\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right)$. $t \in \mathbb{R}$

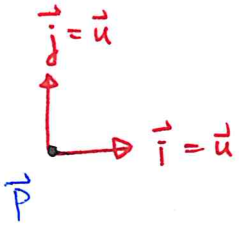
$$D_{\vec{u}} f(\vec{p}) := \lim_{t\to 0} \frac{f(\vec{p}+t\vec{u}) - f(\vec{p})}{t}$$

$$= \lim_{t\to 0} \frac{f\left(1+\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right) - f(1,0)}{t}$$

$$= \lim_{t\to 0} \frac{\left(1+\frac{t}{\sqrt{2}}\right)^2 - 1^2}{t}$$

$$= \lim_{t\to 0} \left(\sqrt{2} + \frac{1}{2}t\right) = \sqrt{2} \quad ※.$$

Special cases:



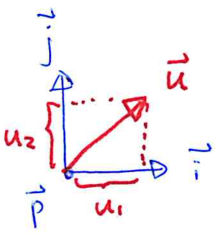
$$D_{\vec{i}} f(\vec{p}) = \frac{\partial f}{\partial x}(\vec{p})$$

$$D_{\vec{j}} f(\vec{p}) = \frac{\partial f}{\partial y}(\vec{p}).$$

(Ex: Check these from definitions.)

Theorem: If f differentiable at $\vec{p}$, then

when $\vec{u} = u_1\vec{i} + u_2\vec{j}$, then



$$\boxed{D_{\vec{u}} f(\vec{p}) = u_1 \frac{\partial f}{\partial x}(\vec{p}) + u_2 \frac{\partial f}{\partial y}(\vec{p})}$$

i.e $\boxed{D_{\vec{u}} f(\vec{p}) = \nabla f(\vec{p}) \cdot \vec{u}}$ —— (*).

where $\nabla f(\vec{p}) := \left(\frac{\partial f}{\partial x}(\vec{p}), \frac{\partial f}{\partial y}(\vec{p})\right)$.

E.g. 2 : Find $D_{\vec{u}} f(\vec{p})$ for $f(x,y) = 2xy - 3y^2$

$\vec{p} = (5,5)$

along $\vec{v} = (4,3)$.

not unit vector.

take $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{(4,3)}{\sqrt{4^2+3^2}} = \left(\frac{4}{5}, \frac{3}{5}\right)$. ← unit vector, $\|\vec{u}\| = 1$

f is differentiable ∵ polynomial.

So formula applies.

$$\left.\frac{\partial f}{\partial x}\right|_{\vec{p}} = \left. 2y \right|_{(5,5)} = 10.$$

$$\left.\frac{\partial f}{\partial y}\right|_{\vec{p}} = \left. 2x - 6y \right|_{(5,5)} = -20.$$

$$D_{\vec{u}} f(\vec{p}) \overset{\text{Thm.}}{=} \nabla f(\vec{p}) \cdot \vec{u} = (10,-20)\cdot\left(\frac{4}{5}, \frac{3}{5}\right) = 8 - 12 = \underline{\underline{-4}}$$

Proof of "$D_{\vec{u}} f(\vec{p}) = \nabla f(\vec{p}) \cdot \vec{u}$"

Remember that:

$$\underbrace{\gamma(t) = \vec{p} + t\vec{u}, \quad t \in \mathbb{R}.}_{\text{line thr. } \vec{p} \text{ at } t=0 \text{ and } /\!/ \text{ to } \vec{u}.}$$

$$D_{\vec{u}} f(\vec{p}) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t))$$

f diff. at $\vec{p}$ $\Rightarrow$ Chain Rule.

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

at $\gamma(0) = \vec{p}$ $t=0$

$\gamma(t) = (x(t), y(t))$
$= (p_1 + tu_1, p_2 + tu_2)$

where $\vec{p} = (p_1, p_2)$
$\vec{u} = (u_1, u_2)$

$$= u_1 \frac{\partial f}{\partial x}(\vec{p}) + u_2 \frac{\partial f}{\partial y}(\vec{p}).$$

$$= \nabla f(\vec{p}) \cdot \vec{u}.$$

∎

Remarks: (1) The formula holds in any dimension.

(2) f is diff. at $\vec{p}$ $\Longrightarrow$ $D_{\vec{u}} f(\vec{p})$ exists for all $\vec{u}$.

not true in general.

A strange example:

Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

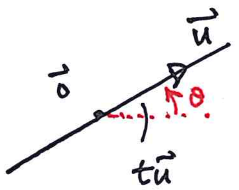
$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0). \end{cases}$$

Claim: (1) f is continuous at $\vec{0}$ (Ex: check this)

(2) $D_{\vec{u}} f(\vec{0})$ exists for all $\vec{u}$.

(3) f is NOT differentiable at $\vec{0}$.

Solution: (2) Along $\vec{u}$, $\|\vec{u}\| = 1$, use definition, (Cannot apply $D_{\vec{u}} f = \nabla f \cdot \vec{u}$ since f is not known to be differentiable)



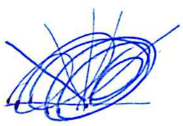
$$D_{\vec{u}} f(\vec{0}) = \frac{d}{dt}\bigg|_{t=0} f(t\vec{u})$$

$\vec{u} = (\cos\theta, \sin\theta)$

$$= \frac{d}{dt}\bigg|_{t=0} f(t\cos\theta, t\sin\theta)$$

[Ex: Visualize the graph of $f(x,y) = z$]

$$= \frac{d}{dt}\bigg|_{t=0} \frac{t^2 \cos\theta \sin\theta}{t}$$



$$= \cos\theta \sin\theta \quad * .$$

(3). From (2), $\left.\dfrac{\partial f}{\partial x}\right|_{\vec{0}} = 0$, $\left.\dfrac{\partial f}{\partial y}\right|_{\vec{0}} = 0$, $f(\vec{0}) = 0$

$(\theta = 0)$ $\quad (\theta = \frac{\pi}{2})$

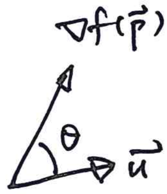
$L(x,y) = 0$ linear approximation.

$$\lim_{(x,y)\to(0,0)} \frac{\varepsilon(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} = \underbrace{\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}}_{\text{limit not exists}}$$

$\overset{?}{=} 0$

f NOT diff. at 0. $\Leftarrow$

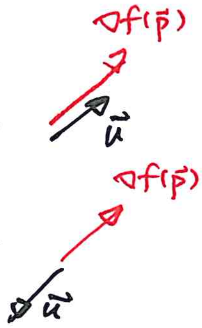
Application: $\displaystyle \max/\min_{\substack{\vec{u}\in\mathbb{R}^2 \\ \|\vec{u}\|=1}} D_{\vec{u}} f(\vec{p})$ related to Physical Problem A.



Recall: Thm $\Rightarrow$ $D_{\vec{u}} f(\vec{p}) = \nabla f(\vec{p}) \cdot \vec{u}$

$$= \|\nabla f(\vec{p})\| \underbrace{\|\vec{u}\|}_{\substack{\shortparallel \\ 1}} \cos\theta$$

$$= \|\nabla f(\vec{p})\| \underbrace{\cos\theta}_{-1 \le \,\cdot\cdot\, \le 1}$$



So, $\max = \|\nabla f(\vec{p})\|$ when $\theta = 0$, ie $\vec{u} \,/\!/\, \nabla f(\vec{p})$

$\min = -\|\nabla f(\vec{p})\|$ when $\theta = \pi$, ie $-\vec{u} \,/\!/\, \nabla f(\vec{p})$.

Warmest fastest

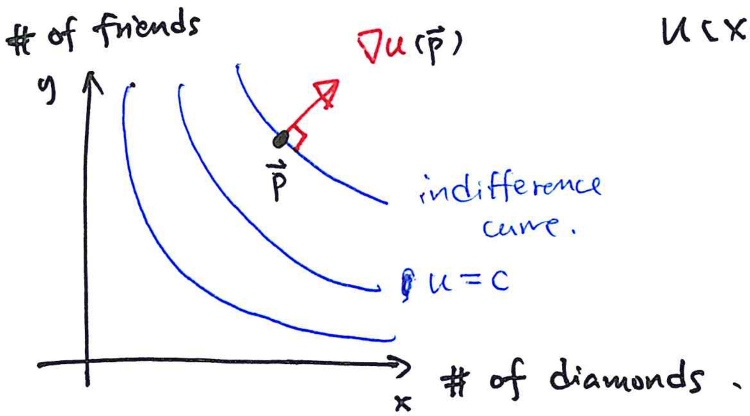
Take $\vec{u} = \dfrac{\nabla f(\vec{p})}{\|\nabla f(\vec{p})\|}$

Coolest fastest

Take $\vec{u} = -\dfrac{\nabla f(\vec{p})}{\|\nabla f(\vec{p})\|}$.

$\boxed{\text{when } \nabla f(\vec{p}) \neq \vec{0}}$

Application in Microeconomics



$u(x,y) =$ level of ☺ given x and y. (utility function)

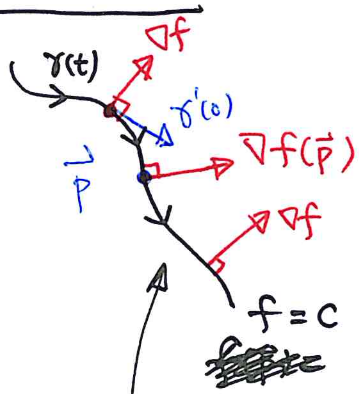
$(= xy \text{ eg.})$

Gradient and Level Sets

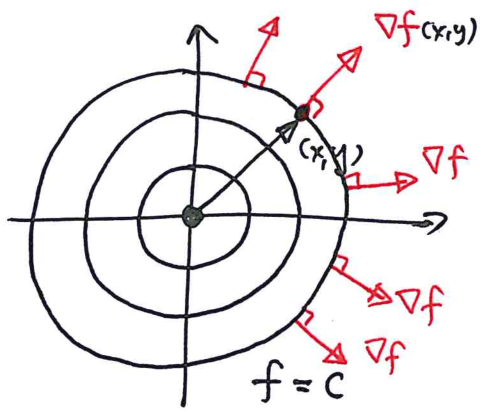
Theorem: f is differentiable and $\nabla f(\vec{p}) \neq \vec{0}$.

$\Rightarrow$ $\nabla f(\vec{p}) \perp$ level set of f passing through $\vec{p}$. (for any dimension n)

$\boxed{\text{Case } n=2}$



E.g. 3: $f(x,y) = x^2+y^2$.



$$\nabla f(x,y) = (2x,2y) = 2(x,y)$$

Idea: parametrize by $\gamma(t): (-1,1) \longrightarrow \mathbb{R}^2$

$$\gamma(0) = \vec{p}$$

$\gamma'(0)$ = velocity at $\vec{p}$
tangent to $\{f=c\}$

Claim: $\boxed{\nabla f(\vec{p}) \perp \gamma'(0)} \Rightarrow \nabla f(\vec{p}) \perp$ to $\{f=c\}$.

Check: $\nabla f(\vec{p}) \cdot \gamma'(0) = 0$

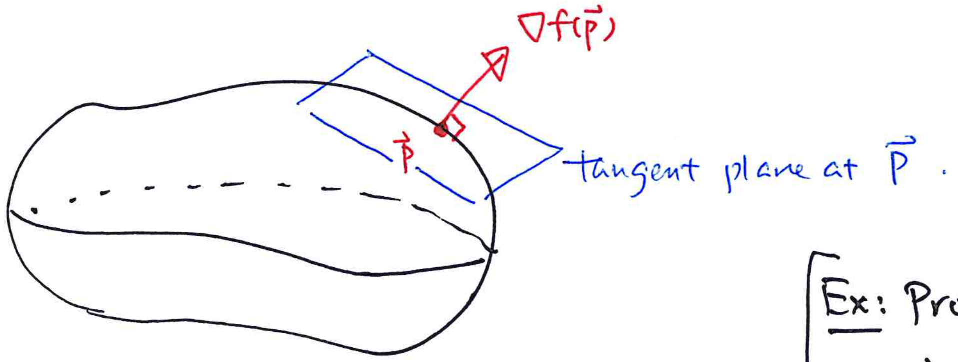
Since $\gamma(t)$ is a curve on $\{f=c\}$.

i.e. $\quad f(\gamma(t)) = c \quad \forall t$

diff. w.r.t t at $t=0$.

$$\underbrace{\nabla f(\vec{p})}_{\perp \text{ to } \{f=c\}} \cdot \underbrace{\gamma'(0)}_{\text{tangent to } \{f=c\}} = 0$$

$\boxed{\text{Case } n=3}$ $f(x,y,z)$. level sets $\boxed{f(x,y,z)=c}$ ← 2D.



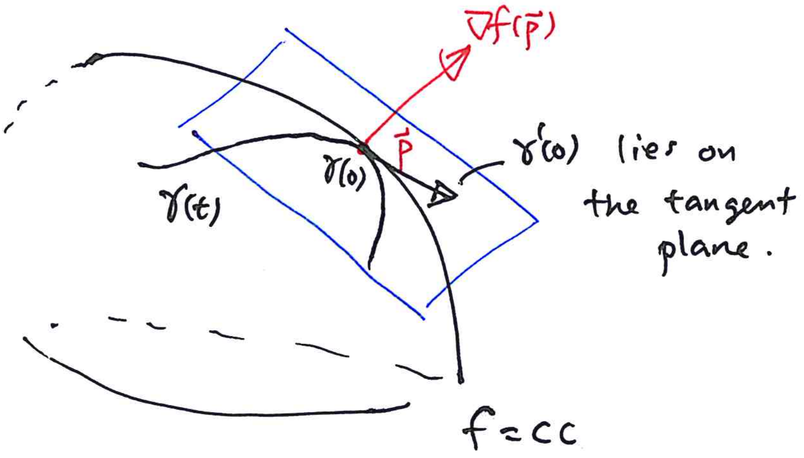
$f=c$ level surface.

[Ex: Prove the thm in $n=3$]

Idea: Take any curve $\gamma(t) : (-1,1) \longrightarrow \mathbb{R}^3$

s.t $\gamma(0) = \vec{p}$

and $\gamma'(0)$ = tangent to the level surface $\{f=c\}$.



show that

$$\underset{\substack{\Downarrow \\ \text{normal to the level surface.}}}{\nabla f(\vec{p})} \cdot \underbrace{\gamma'(0)}_{\text{any tangent vector}} = 0$$